Resonant homoclinic flip bifurcations: a numerical investigation

Bernd Krauskopf, Bart E. Oldeman, and Alan R. Champneys
Engineering Mathematics, University of Bristol,
Bristol BS8 1TR, UK
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Abstract. We report on a numerical study of codimension-three resonant inclination flip homoclinic orbits. Two cases of unfoldings are presented: one with a single homoclinic doubling, and the other containing a homoclinic-doubling cascade. Our results confirm the unfoldings given in Ref. [3].

We consider a homoclinic orbit to a hyperbolic equilibrium, the origin for convenience, in a three-dimensional vector field. The origin is assumed to have two stable and one unstable eigenvalue, which can be scaled to be $-\lambda_{ss}$, $-\lambda_s$ and $\lambda_u = 1$ with $-\lambda_{ss} < -\lambda_s < 0$. Our interest is in homoclinic orbits of higher codimension. One speaks of an *orbit flip* if the homoclinic orbit approaches the origin along the strong stable manifold W^{ss}, and of an *inclination flip* if the stable manifold $W^{ss,s}$ intersects the manifold $W^{s,u}$ nontransversely along the homoclinic orbit. In both bifurcations $W^{ss,s}$ changes orientation. These homoclinic flip bifurcations are of codimension-two, provided certain non-resonance conditions on λ_{ss} and λ_s are satisfied; for details see [3] and references therein. Depending on λ_{ss} and λ_s there are three different cases: **A** where no extra bifurcations occur, **B** with a single homoclinic-doubling bifurcation, and the complicated case **C** involving n-periodic and n-homoclinic orbits for arbitrary n and a region with horseshoe dynamics. If a non-resonance conditions for λ_{ss} and λ_s fails, we speak of a *resonant homoclinic flip bifurcation.* This constitutes a transition between two codimension-two homoclinic flip bifurcations, and there are two such transitions, namely from **A** to **B** and from **B** to **C**. Assuming cone structure close to the codimension-three central singularity, unfoldings for resonant homoclinic flip bifurcations were given in [3] on a sphere around this singularity.

It is the aim of this paper to study these unfoldings numerically. To this end we computed bifurcation curves in the model of Sandstede [4] (which was constructed to contain homoclinic flip bifurcations) with the continuation package AUTO/HOMCONT [1]. We work on spheres in the 'world map' representation

$$\nu_1 = r\cos\pi\phi\cos\pi\theta\,, \quad \nu_2 = r\sin\pi\theta\,, \quad \nu_3 = r\sin\pi\phi\cos\pi\theta, \tag{1}$$

where $\theta \in [-0.5, 0.5]$, $\phi \in [0, 2]$, $r \in \mathbb{R}^+$ and (ν_1, ν_2, ν_3)-space is the original parameter space with the resonant homoclinic flip bifurcation point at the origin. Note that (ν_1, ν_2, ν_3) needs to be matched for each case to the unfolding parameters in Sandstede's model, the details of which will not be given here. We present the results for the inclination flip; those for the orbit flip are similar. A detailed exposition of all of our results will appear elsewhere.

The transition from **A** *to* **B**. The central singularity for this transition is an inclination flip bifurcation with the resonance $\lambda_s = 1$. In the spherical coordinates (1) for a suitable

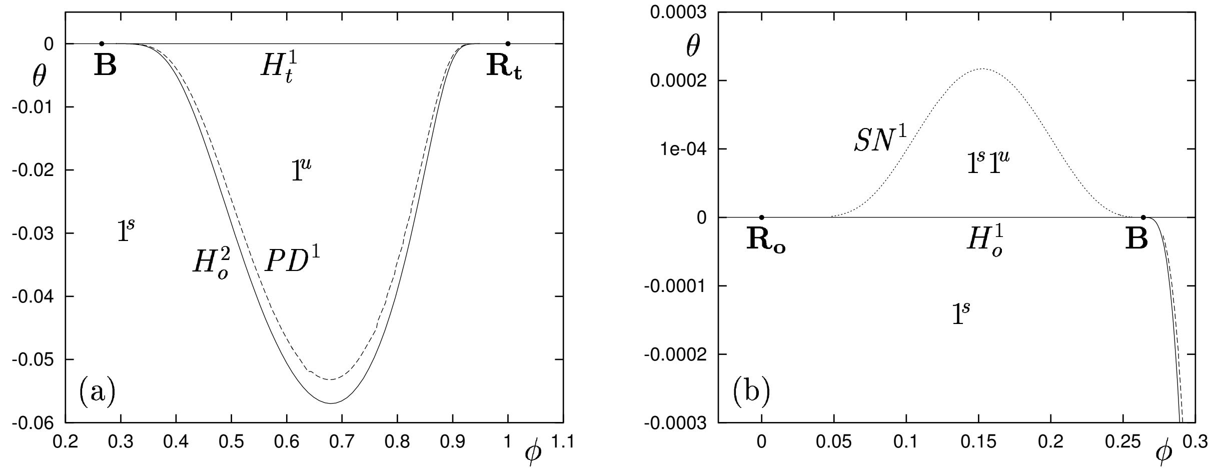

Figure 1: The numerically computed transition from **A** to **B** for the inclination flip for $r = 0.25$ (a) and $r = 0.4$ (b); notice the difference in scale.

choice of (ν_1, ν_2, ν_3) we obtained the numerical results in figure 1. The basic 1-homoclinic orbit changes from oriented to twisted at the points **A** and **B**. Note that at **A** there are no extra bifurcations, which is why it is not shown in figure 1. For a radius of $r = 0.25$ the period doubling curve PD^1 and 2-homoclinic curve H_o^2 were calculated in the (ϕ, θ)-plane in panel (a). These curves PD^1 and H_o^2 connect **B** with a resonant homoclinic point $\mathbf{R_t}$ of a twisted homoclinic orbit. In this and the next figure limit cycles are indicated by their basic period, together with a superscript showing whether they are stable (s) or not (u). It turns out that for $r = 0.25$ the saddle-node curve (known to exist near **B**) is so close to the main homoclinic curve H_o^1 in parameter space that it cannot be followed, and instead we produced panel (b) for $r = 0.4$. The curve SN^1 connects **B** with a resonant homoclinic point $\mathbf{R_o}$ of an oriented homoclinic orbit. Together panels (a) and (b) confirm the unfolding in [3]. Notice that even for $r = 0.4$ the curve SN^1 is very close to the curve H_o^1. It turns out that for $r = 0.4$, the curves PD^1 and H_o^2 are involved in extra bifurcations, so that there is a clear trade-off: when the radius r of the sphere is too small one is not able to resolve curves, but when it is too large extra bifurcation structures appear on the sphere.

The transition from **B** *to* **C**. The central singularity we consider for this transition is an inclination flip bifurcation with the resonance $\lambda_s = 1/2$ for $\lambda_{ss} > 1$. In the spherical coordinates (1) for a suitable choice of (ν_1, ν_2, ν_3) we obtained the numerical results in the (ϕ, θ)-plane for $r = 0.2$ shown in figure 2. The 2-homoclinic curve $H_{o/t}^2$ emerging at **B**, changes its orientation in an inclination flip bifurcation (of type **B**), and can be followed to **C**. From there a 4-homoclinic curve $H_{o/t}^4$ emerges, changes its orientation in another inclination flip bifurcation, and can also be followed to **C**, and so on. In other words, by following H_o^2 we encounter a homoclinic-doubling cascade, which was recently shown to exist near the codimension-three bifurcation studied here [2, 3]. At each of the inclination flip points a period-doubling curve of the respective multiple of the basic period emerges. The curve PD^1 starting at **B** can be continued to **C**, and the same is true for the curves

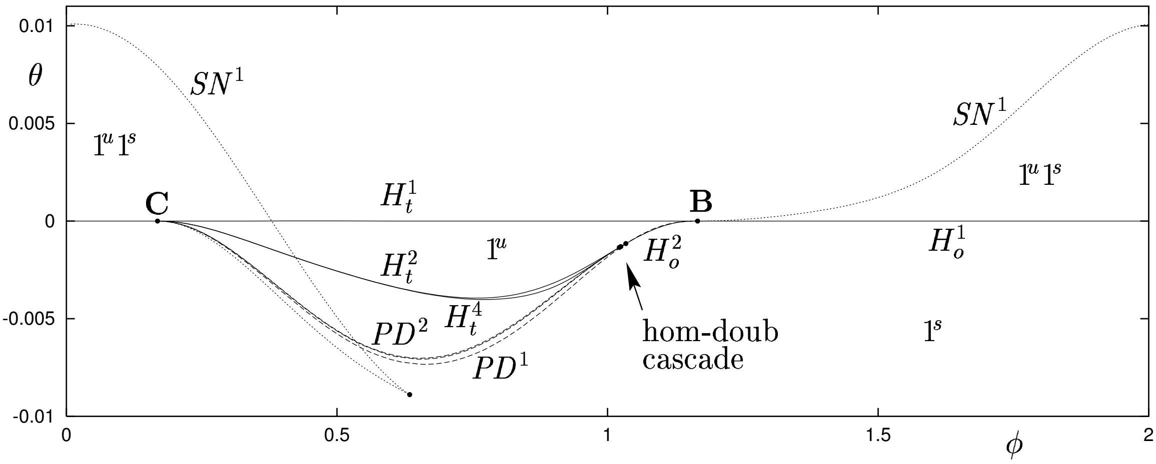

Figure 2: The numerically computed transition from **B** to **C** for the inclination flip involving a homoclinic-doubling cascade.

PD^2 and PD^4 emerging from the inclination flips of the homoclinic-doubling cascade. The saddle-node curve SN^1 starting at **B** continues through a cusp to **C**, whereas the saddle-node bifurcation curves emerging from the inclination flips of the homoclinic-doubling cascade (which are too small to be visible in figure 2) connect to period-doubling curves PD^{2^n}, which then cross the curves $H_o^{2^{n+1}}$ [2]. Figure 2 confirms the unfolding for the transition from **B** to **C** in [3]. The banana-shaped region between the homoclinic and the period-doubling curves is known to contain complicated dynamics [3], namely 'bubbles' of homoclinic-doubling cascades with any odd base period and horseshoe dynamics.

Conclusions. The unfoldings presented in [3] were numerically found and computed in the three-dimensional vector field model from [4] for the case that the central singularity is a resonant inclination flip. A homoclinic-doubling cascade was found numerically as predicted by the theory. The bifurcation structures discussed here, in particular homoclinic-doubling cascades, are expected to be of importance in models from applications with resonant homoclinic flip bifurcations. A more detailed presentation of all cases of transitions and of the homoclinic-doubling cascade is in progress.

References.

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